Lecture #4:
The Spectrum
A consistent estimator is one which converges to the true value when the amount of data increases to infinity.
A stationary process is linear ergodic if the sample mean of one (longer and longer) realization is a consistent estimator of the mean.
A stationary process is quadratic ergodic if the covariance estimator applied to one realization is consistent.
A stationary process is ergodic if every estimator consistent on an ensemble is also consistent on one realization.
Note: only stationary processes can be ergodic.
### Averaging over realizations

\[
\hat{m}(t) = \frac{1}{N} \sum_{i=1}^{N} X_i(t)
\]

- **Works always**
- **Many realizations needed**

### Averaging over time

\[
\hat{m}(t) = \frac{1}{n} \sum_{s=1}^{n} X(s)
\]

- **Only one (long) realization is needed**
- **Ergodic stationary processes only**

**Repetition**
Averaging over realizations

\[ \hat{m}(t) = \frac{1}{N} \sum_{i=1}^{N} X_i(t) \]

\[ \mathbb{E}[\hat{m}(t)] = m(t) \]

\[ \mathbb{V}[\hat{m}(t)] = \frac{1}{N} \sigma^2(t) \]

Consistent

Averaging over time

\[ \hat{m}(t) = \frac{1}{n} \sum_{s=1}^{n} X(s) \]

\[ \mathbb{E}[\hat{m}(t)] = m(t) \]

\[ \mathbb{V}[\hat{m}] = \frac{1}{n^2} \sum_{\tau=-n+1}^{n-1} (n - |\tau|) r_X(\tau) \]

Consistent if ergodic
If you don’t know the mean function, estimate it first! Then, you estimate covariance function by:

\[
\hat{r}(\tau) = \begin{cases} 
\frac{1}{n} \sum_{t=1}^{n-\tau} (x(t) - m)(x(t + \tau) - m) & 0 \leq \tau \leq n - 1 \\
\hat{r}(-\tau) & -n + 1 \leq \tau \leq -1 \\
0 & \text{otherwise}
\end{cases}
\]

This estimator is biased, but asymptotically the bias goes to zero. Note: Obviously, this estimator is only useful for stationary processes!
• The covariance function contains interesting information

• Spectrum: the Fourier transform of the covariance function - reveals information
In many applications, the interesting information is described by the covariance function of a stationary random process.
• Recorded piano

• Estimated covariance function

• Another process with the same covariance function

* Audio content lost during conversion from Keynote to PDF.
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Realizations & covariance functions

Realization

Covariance function
The spectrum (or spectral density) of a stationary process is the Fourier transform of the covariance function.
Help Mr Rabbit to find his match!
Help Mrs Crocodile to find her match!
<table>
<thead>
<tr>
<th>r(τ) α &gt; 0</th>
<th>( R(f) = \int_{-\infty}^{\infty} e^{-i2\pi f \tau} r(\tau) , d\tau )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( e^{-\alpha</td>
<td>\tau</td>
</tr>
<tr>
<td>( e^{-\alpha</td>
<td>\tau</td>
</tr>
</tbody>
</table>
Covariance functions & spectra

Covariance function

\[ r_x(\tau) \]

\[ \alpha = 0.5, f_0 = 0.1 \]

\[ r_x(\tau) \]

\[ \alpha = 0.5, f_0 = 0.2 \]

\[ r_x(\tau) \]

\[ \alpha = 0.5, f_0 = 1 \]

Spectral density

\[ R_x(f) \]

\[ f \leq 2 \]
Which one matches?
The answer
A piano tone
The covariance function
The spectrum
Make your own spectrum analyzer

Useful Matlab commands:

```plaintext
>> help wavread
>> help resample
>> help fft
>> help fftshift
```
Let \( f_1, f_2, \ldots, f_n \) be positive real numbers. Let \( A_1, A_2, \ldots, A_n \) be independent r.v. and let \( \phi_1, \phi_2, \ldots, \phi_n \) be independent r.v. with uniform distribution in \((0, 2\pi]\). Define a random process, \( \{X(t), t \in \mathbb{R}\} \) by

\[
X(t) = \sum_{k=1}^{n} A_k \cos(2\pi f_k + \phi_k).
\]

Then, 

\[
r_X(\tau) = \sum_{k=1}^{n} E[A_k^2] \cos(2\pi f_k \tau).
\]

Properties of the covariance function of a weakly stationary random process:

- \( r(0) = \mathbb{E}[X(t)] \geq 0 \)
- \( r(\tau) = r(-\tau) \)
- \( r(0) \geq |r(\tau)| \)
- \( r \) is continuous at zero \( \iff \) \( r \) is continuous everywhere
- \( r \) is non-negative definite:
  \[
  \sum_{i} \sum_{j} \alpha_i \alpha_j r(t_i - t_j) \geq 0
  \]

A process \( \{X(t), t \geq 0\} \) is a Poisson process with parameter \( \lambda \) if

- \( X(0) = 0 \), and
- it has stationary and independent increments,
- the increment \( X(t+h) - X(t) \) is Poisson distributed with mean \( \lambda h \).

\[
\sim \text{Po}(\lambda t_1) \sim \text{Po}(\lambda t_2) \sim \text{Exp}(\lambda - 1)t_1t_2
\]

\[
R(f) \]

\[
\text{Energy}
\]

\[
\text{Frequency (Hz)}
\]

Low frequency
Let \( f_1, f_2, \ldots, f_n \) be positive real numbers.

Let \( A_1, A_2, \ldots, A_n \) be independent r.v. and let \( \phi_1, \phi_2, \ldots, \phi_n \) be independent r.v. with uniform distribution in \((0, 2\pi]\). Define a random process, \( \{X(t), t \in \mathbb{R}\} \) by

\[
X(t) = \sum_{k=1}^{n} A_k \cos(2\pi f_k t + \phi_k).
\]

Then, \( r_X(\tau) = \frac{1}{2} \sum_{k=1}^{n} \mathbb{E}[A_k^2] \cos(2\pi f_k \tau) \).

Properties of the covariance function of a weakly stationary random process:

- \( r(0) = \mathbb{V}[X(t)] \geq 0 \)
- \( r(\tau) = r(-\tau) \)
- \( r(0) \geq |r(\tau)| \)
- \( r \) is continuous at zero \( \iff \) \( r \) is continuous everywhere
- \( r \) is non-negative definite:
  \[
  \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \alpha_j r(t_i - t_j) \geq 0
  \]

A process \( \{X(t), t \geq 0\} \) is a Poisson process with parameter \( \lambda \) if

- \( X(0) = 0 \), and
- it has stationary and independent increments,
- the increment \( X(t+h) - X(t) \) is Poisson distributed with mean \( \lambda h \).

\( \sim \text{Po}(\lambda t_1) \sim \text{Po}(\lambda t_2) \)

\( \sim \text{Exp}(\lambda t_1 t_2) \)

\( \mathcal{R}(f) \)
Early course evaluation

Please write down some of your thoughts on

• The exercises
• The lectures
• The course in general

Leave your answers anonymously in the envelope.