On Estimation
<table>
<thead>
<tr>
<th>Monday</th>
<th>Tuesday</th>
<th>Wednesday</th>
<th>Thursday</th>
<th>Friday</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td></td>
<td></td>
<td>Exercise MH:309B</td>
<td>April 15:</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Except: April 15</td>
<td>Lecture M:B</td>
</tr>
<tr>
<td>9</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>Lab 2,3</td>
<td>Lab 1</td>
<td>Exercise MH:362B</td>
<td></td>
</tr>
<tr>
<td>11</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>12</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>Lecture M:B</td>
<td>Exercise</td>
<td>Lecture M:B</td>
<td>Lab 1</td>
</tr>
<tr>
<td></td>
<td></td>
<td>MH:362B</td>
<td></td>
<td></td>
</tr>
<tr>
<td>14</td>
<td>Lab 3</td>
<td>Lab 2</td>
<td></td>
<td>Lab 1</td>
</tr>
<tr>
<td>15</td>
<td>Exercise</td>
<td></td>
<td></td>
<td>April 16:</td>
</tr>
<tr>
<td></td>
<td>MH:309B</td>
<td></td>
<td></td>
<td>Exer.</td>
</tr>
<tr>
<td>16</td>
<td></td>
<td></td>
<td></td>
<td>Lab 1</td>
</tr>
</tbody>
</table>
Properties of the covariance function of a weakly stationary random process:

- $r(0) = \mathbb{V}[X(t)] \geq 0$
- $r(\tau) = r(-\tau)$
- $r(0) \geq |r(\tau)|$
- $r$ is continuous at zero $\iff r$ is continuous everywhere
- $r$ is non-negative definite: $\sum_i \sum_j \alpha_i \alpha_j r(t_i - t_j) \geq 0$
A process \( \{X(t), \ t \geq 0\} \) is a Poisson process with parameter \( \lambda \) if

- \( X(0) = 0 \), and

- it has stationary and independent increments, and

- the increment \( X(t+h) - X(t) \) is Poisson distributed with mean \( \lambda h \).
Definition. Let $T$ be a subset of the real line, $T \subseteq \mathbb{R}$. A set of random variables \( \{X_t, t \in T\} \) is called a random (or stochastic) process with time domain $T$.

Definition. Let \( \{X_t, t \in T\} \) be a random process. The function \( m_{X}: T \mapsto \mathbb{R} \) given by \( m_{X}(t) = \mathbb{E}[X_t] \) is called the mean function of the random process \( \{X_t, t \in T\} \) (if the expectation exists).

Definition. Let \( \{X_t, t \in T\} \) be a random process. The function \( r_{X}: T^2 \mapsto \mathbb{R} \) given by \( r_{X}(s, t) = \mathbb{C}[X_s, X_t] \) is called the covariance function of the random process \( \{X_t, t \in T\} \) (if the covariance exists).

Definition. A random process \( \{X_t, t \in T\} \) is weakly stationary if, and only if, both of the following two conditions are satisfied:

- The mean function is constant, that is, there exist a constant $c$ such that $m_{X}(t) = c \forall t \in T$.
- The covariance function depends only on the time-lag, that is, there exist a function $\tilde{r}_{X}$ such that:
  \[ r_{X}(s, t) = \tilde{r}_{X}(s - t) \forall (s, t) \in T^2. \]
  If such a function exists, then $\tilde{r}_{X}$ is also called the covariance function of the process.

Let $Y$ and $Z$ be two independent random variables with expectation zero and variance $\sigma^2_Y$ and $\sigma^2_Z$. We construct the random process \( \{X(t), t \in \mathbb{R}\} \) by:
\[
X(t) = Y \sin(t) + Z.
\]

Is this process weakly stationary?

Let $A \in \mathcal{U}(0, 1)$ and $\phi \in \mathcal{U}(0, 2\pi)$ be two independent random variables. Define the random process \( \{X(t), t \in \mathbb{R}\} \) by:
\[
X(t) = A \cos(t + \phi).
\]

Is this process weakly stationary?

Definition. Let \( \{X(t), t \in T\} \) be any random process. The function \( r_{X}(s, t) = \mathbb{C}[X(s), X(t)] \) is called the covariance function.

Definition. Let \( \{X(t), t \in T\} \) be a weakly stationary random process. The function \( r_{X}(\tau) = \mathbb{C}[X(t), X(t + \tau)] \) is called the covariance function.

Let $r$ be the covariance function of a weakly stationary random process. Prove that $r(0) \geq |r(\tau)| \forall \tau$.
Is this process weakly stationary?

Let \( X_t \) be a random process. The function \( \tau \to X_t(\tau + \omega) - X_t(\omega) \) is called the covariance function.

A set of random variables \( Y_t \) is called a stationary process if, and only if, both of the following two conditions are satisfied:

1. \( \lambda \in \mathbb{R} \)
   - \( \lambda = 0 \): \( \mathbb{E}[Y_t] = \text{constant} \)
   - \( \lambda = 1 \): \( \text{Var}(Y_t) = \text{constant} \)
   - \( \lambda = 2 \): \( \text{Cov}(Y_t, Y_{t+\lambda}) = \text{constant} \)

\( \tau \): time

\( Y_t \): random variable

\( X_t \): random process

\( \lambda \): constant

\( \mathbb{E} \): expectation function

\( \text{Var} \): variance function

\( \text{Cov} \): covariance function

\( \mathbb{R} \): set of real numbers
Definition. Let $T$ be a subset of the real line, $T \subseteq \mathbb{R}$. A set of random variables $\{X_t, t \in T\}$ is called a random (or stochastic) process with time domain $T$.

Definition. Let $\{X_t, t \in T\}$ be a random process. The function $m_{X_t} : T \mapsto \mathbb{R}$ given by $m_{X_t}(t) = \mathbb{E}[X_t]$ is called the mean function of the random process $\{X_t, t \in T\}$ (if the expectation exists).

Definition. Let $\{X_t, t \in T\}$ be a random process. The function $r_{X_t} : T^2 \mapsto \mathbb{R}$ given by $r_{X_t}(s, t) = \mathbb{C}[X_s, X_t]$ is called the covariance function of the random process $\{X_t, t \in T\}$ (if the covariance exists).

Definition. A random process $\{X_t, t \in T\}$ is weakly stationary if, and only if, both of the following two conditions are satisfied:

1. The mean function is constant, that is, there exist a constant $c$ such that $m_{X_t}(t) = c \quad \forall t \in T$.
2. The covariance function depends only on the time-lag, that is, there exist a function $\tilde{r}_{X_t}$ such that $r_{X_t}(s, t) = \tilde{r}_{X_t}(s - t) \quad \forall (s, t) \in T^2$. If such a function exists, then $\tilde{r}_{X_t}$ is also called the covariance function of the process.

Let $Y$ and $Z$ be two independent random variables with expectation zero and variance $\sigma_Y^2$ and $\sigma_Z^2$. We construct the random process $\{X_t, t \in \mathbb{R}\}$ by:

$$X_t = Y \sin(t) + Z.$$ 

Is this process weakly stationary?

Let $A \in \mathcal{U}(0, 1)$ and $\phi \in \mathcal{U}(0, 2\pi)$ be two independent random variables. Define the random process $\{X_t, t \in \mathbb{R}\}$ by:

$$X_t = A \cos(t + \phi).$$ 

Is this process weakly stationary?
• Estimation of the mean function
• Ergodic processes
• Estimation of the covariance function
Let $e_0, e_{\pm 1}, e_{\pm 2}, \ldots$ be i.i.d. r.v. with zero mean and unit variance. Define the random process $\{X_t, t \in \mathbb{Z}\}$ by:

$$X_t = 7 + e_t + 0.6e_{t-1} - 0.3e_{t-2}$$
n = 10000;
e = randn(n,1);
X = 7 + ... 
    filter([1 0.6 -0.3], 1, e);
plot(X);
Here are four realizations of this process.

Assuming, that we don’t know how the process was generated, how should we estimate the mean function?
We consider $X_1(t), X_2(t), X_3(t), X_4(t)$, to be i.d. observations from a r.v. with mean $m(t)$ and variance $\sigma^2(t)$. 
\[ \hat{m}(t) = \]

<table>
<thead>
<tr>
<th>(X_1(t))</th>
<th>(X_2(t))</th>
<th>(X_3(t))</th>
<th>(X_4(t))</th>
</tr>
</thead>
<tbody>
<tr>
<td>6.06</td>
<td>4.84</td>
<td>7.59</td>
<td>7.62</td>
</tr>
</tbody>
</table>
\[ \hat{m}(t) = \frac{1}{4} \sum_{i=1}^{4} X_i(t) \approx 6.53 \]

- \[ X_1(t) = 6.06 \]
- \[ X_2(t) = 4.84 \]
- \[ X_3(t) = 7.59 \]
- \[ X_4(t) = 7.62 \]
\[ \hat{m}(t) = \frac{1}{4} \sum_{i=1}^{4} X_i(t) \approx 6.53 \]

\[ \mathbb{E} [\hat{m}(t)] = \]

\[ X_1(t) = 6.06 \]
\[ X_2(t) = 4.84 \]
\[ X_3(t) = 7.59 \]
\[ X_4(t) = 7.62 \]
\[ \hat{m}(t) = \frac{1}{4} \sum_{i=1}^{4} X_i(t) \approx 6.53 \]

\[ \mathbb{E}[\hat{m}(t)] = m(t) \]

\[ X_1(t) = 6.06 \]
\[ X_2(t) = 4.84 \]
\[ X_3(t) = 7.59 \]
\[ X_4(t) = 7.62 \]
\[ \hat{m}(t) = \frac{1}{4} \sum_{i=1}^{4} X_i(t) \approx 6.53 \]

\[ \mathbb{E} [\hat{m}(t)] = \hat{m}(t) \]

\[ \text{V} [\hat{m}(t)] = s^2 = \]

\[ X_1(t) = 6.06 \]
\[ X_2(t) = 4.84 \]
\[ X_3(t) = 7.59 \]
\[ X_4(t) = 7.62 \]
\[
\hat{m}(t) = \frac{1}{4} \sum_{i=1}^{4} X_i(t) \approx 6.53
\]

\[
E[\hat{m}(t)] = m(t)
\]

\[
V[\hat{m}(t)] = s^2 = \frac{1}{4} \sigma^2(t)
\]

Is the estimate consistent?

<table>
<thead>
<tr>
<th>Function</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>(X_1(t))</td>
<td>6.06</td>
</tr>
<tr>
<td>(X_2(t))</td>
<td>4.84</td>
</tr>
<tr>
<td>(X_3(t))</td>
<td>7.59</td>
</tr>
<tr>
<td>(X_4(t))</td>
<td>7.62</td>
</tr>
</tbody>
</table>
\[ \hat{m}(t) = \frac{1}{4} \sum_{i=1}^{4} X_i(t) \approx 6.53 \]

\[ \mathbb{E}[\hat{m}(t)] = m(t) \]

\[ \mathbb{V}[\hat{m}(t)] = s^2 = \frac{1}{4} \sigma^2(t) \]

Is the estimate consistent? Yes!

\[ \hat{s} = \]

\[ X_1(t) = 6.06 \]
\[ X_2(t) = 4.84 \]
\[ X_3(t) = 7.59 \]
\[ X_4(t) = 7.62 \]
Mean function estimation

\[
\hat{m}(t) = \frac{1}{4} \sum_{i=1}^{4} X_i(t) \approx 6.53
\]

\[
E[\hat{m}(t)] = m(t)
\]

\[
V[\hat{m}(t)] = s^2 = \frac{1}{4} \sigma^2(t)
\]

Is the estimate consistent? Yes!

\[
\hat{s} = \sqrt{\frac{1}{4} \frac{1}{3} \sum_{i=1}^{4} (X_i(t) - \hat{m})^2} \approx 0.67
\]

\[
I_{\hat{m}(t)} \approx \hat{m}(t) \pm 2 \cdot \hat{s} \approx [5.19, 7.87]
\]

\[
X_1(t) = 6.06
\]

\[
X_2(t) = 4.84
\]

\[
X_3(t) = 7.59
\]

\[
X_4(t) = 7.62
\]

If \( \hat{m}(t) \) is Gaussian.
Repeat this procedure for all time points!

* By mistake, the mean ±4 standard deviations (rather than ±2) has been plotted. By yet another mistake I lost my simulated noise process which makes re-plotting almost impossible...
We consider $X(1), X(2), \ldots, X(n)$ to be correlated observations from a r.v. with mean $m(t) = m$. If we assume that the process is stationary, we can estimate the mean using only one realization!
\( \hat{m} = \)
\[ \hat{m} = \frac{1}{n} \sum_{t=1}^{n} X(t) \approx 7.26 \]

\[ \mathbb{E}[\hat{m}] = \]

Mean function estimation
\[ \hat{m} = \frac{1}{n} \sum_{t=1}^{n} X(t) \approx 7.26 \]

\[ \mathbb{E} [\hat{m}] = m \]

\[ \mathbb{V} [\hat{m}] = \]
\[ \hat{m} = \frac{1}{n} \sum_{t=1}^{n} X(t) \approx 7.26 \]

\[ \mathbb{E} [\hat{m}] = m \]

\[ \mathbb{V} [\hat{m}] = \frac{1}{n^2} \sum_{\tau=-n+1}^{n-1} (n - |\tau|) r_X(\tau) \approx \frac{1}{n} \sum_{\tau=-\infty}^{\infty} r_X(\tau) \]

Is the estimate consistent?
\[ \hat{m} = \frac{1}{n} \sum_{t=1}^{n} X(t) \approx 7.26 \]

\[ \mathbb{E} [\hat{m}] = m \]

\[ \text{Var} [\hat{m}] = \frac{1}{n^2} \sum_{\tau=-n+1}^{n-1} (n - |\tau|) r_X(\tau) \approx \frac{1}{n} \sum_{\tau=-\infty}^{\infty} r_X(\tau) \]

Is the estimate consistent? Only if the process is (linear) ergodic!
It is, since:

\[ r_X(\tau) = \begin{cases} 
1.45 & \tau = 0 \\
0.42 & \tau = \pm 1 \\
-0.3 & \tau = \pm 2 \\
0 & \text{otherwise}
\end{cases} \]

And therefore, the variance:

\[
V[\hat{m}] = \frac{1}{n^2} \sum_{\tau=-n+1}^{n-1} (n - |\tau|) r_X(\tau) \approx \frac{1}{n} \sum_{\tau=-\infty}^{\infty} r_X(\tau)
\]

will go to zero when \( n \) goes to infinity.

Later we will discuss how to estimate this!
So, our process is ergodic, and the estimate converges to the correct value.
Let $e_0, e_{\pm 1}, e_{\pm 2}, \ldots$ be i.i.d. r.v. with zero mean and unit variance and let $M \in N(7, 7^2)$ be a Gaussian random variable. Define the random process $\{X_t, t \in \mathbb{Z}\}$ by:

$$X_t = M + e_t + 0.6e_{t-1} - 0.3e_{t-2}$$
n = 10000;
e = randn(n,1);
M = normrnd(7,7);
X = M + ...
    filter([1 0.6 -0.3], 1, e);
plot(X);
Here are four realizations of this process.

Is it ergodic?
It is not, since:

\[ r_X(\tau) = \begin{cases} 
50.45 & \tau = 0 \\
49.42 & \tau = \pm 1 \\
48.7 & \tau = \pm 2 \\
49 & \text{otherwise}
\end{cases} \]

And therefore, the variance:

\[ V[\hat{m}] = \frac{1}{n^2} \sum_{\tau=-n+1}^{n-1} (n - |\tau|) r_X(\tau) \]

will not go to zero when \( n \) goes to infinity.
So, if we just have one realization the estimate of the mean would not converge to the true value, no matter how many observations we have.
Consider a realization of the ergodic process.

Let’s estimate the covariance function!
\[
\hat{r}(\tau) = \frac{1}{n} \sum_{t=1}^{n-\tau} \left(x(t) - m\right) \left(x(t + \tau) - m\right)
\]
\[ \hat{r}(\tau) = \frac{1}{n} \sum_{t=1}^{n-\tau} \left( x(t) - m \right) \left( x(t + \tau) - m \right) \]
Use the biased covariance function estimator!

Reason: It is non-negative definite, it has lower mean square error and less risk of spurious estimates for large time-lags.
If the non-biased estimator was used, the result would have been much poorer:
Assuming that the AR-process is stationary, we proved that the mean is zero, and we discussed how the covariance function could be determined by the Yule-Walker equations:

\[ r_X(\tau) + a_1 r_X(\tau - 1) + \cdots + a_p r_X(\tau - p) = \begin{cases} \sigma^2 & \tau = 0 \\ 0 & \tau \neq 0 \end{cases} \]

Assuming stationarity, we derived the spectral density:

\[ R_X(f) = \frac{\sigma^2}{\sum_{k=0}^{p} a_k e^{-i2\pi fk}} \]

Assume that the stationary random process \( \{X(t), t \in \mathbb{Z}\} \) has covariance function

\[ r_X(\tau) = \begin{cases} 4 & \tau = 0 \\ 2 & \tau = \pm 1 \\ -1 & \tau = \pm 2 \\ 0 & \text{otherwise} \end{cases} \]

Consider the following estimators of the mean:

\[ \hat{m}_1 = \frac{X(t) + X(t-1)}{2}, \quad \hat{m}_2 = \frac{X(t) + X(t-2)}{2}, \quad \hat{m}_3 = \frac{X(t) + X(t-3)}{2}. \]

Which one is the best?
Consider the following covariance function:

\[ r(\tau) = \frac{2 + \tau^2}{1 + \tau^2} \]

Can it be the covariance function of an ergodic process?
• Estimation of the mean function

• Ergodic processes

• Estimation of the covariance function