FOUR EASY QUESTIONS:

★ What is a Random Process?
★ What is the Mean Function?
★ What is the Covariance Function?
★ What is Weakly Stationary?

THIS LECTURE: THE POISSON PROCESS + WHAT I FORGOT TO TELL YOU LAST TIME!
Definition. Let $T$ be a subset of the real line, $T \subseteq \mathbb{R}$. A set of random variables $\{X_t, t \in T\}$ is called a random (or stochastic) process with time domain $T$. 
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Say “Hello” to Random Processes!

Take The Quiz

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**Definition.** Let \( \{X_t, t \in T\} \) be a random process. The function \( m_X : T \rightarrow \mathbb{R} \) given by

\[
m_X(t) = \mathbb{E}[X_t]
\]

is called the *mean function* of the random process \( \{X_t, t \in T\} \) (if the expectation exists).
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THE SECOND LECTURE

TAKE THE QUIZ

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Definition. Let \( \{X_t, t \in T\} \) be a random process. The function \( r_X : T^2 \mapsto \mathbb{R} \) given by

\[
r_X(s, t) = C[X_s, X_t]
\]

is called the **covariance function** of the random process \( \{X_t, t \in T\} \) (if the covariance exists).
Say "Hello" to Random Processes!

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TAKE THE QUIZ

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Definition. A random process \( \{X_t, t \in T\} \) is \textit{weakly stationary} if, and only if, both of the following two conditions are satisfied:
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- The covariance function depends only on the time-lag, that is, there exist a function \( \tilde{r} \) such that: \( r_X(s, t) = \tilde{r}(s - t) \forall (s, t) \in T^2 \). If such a function exists, then \( \tilde{r}_X \) is also called the covariance function of the process.
• Examples of Random Processes

• More on the Covariance Function

• The Poisson Process
Let $Y$ and $Z$ be two independent random variables with expectation zero and variance $\sigma_Y^2$ and $\sigma_Z^2$. We construct the random process \{\(X(t), t \in \mathbb{R}\)\} by:

$$X(t) = Y \sin(t) + Z.$$ 

Is this process weakly stationary?
Example 1

Y = randn;
Z = rand - 0.5;
X = Y*sin([-8:0.01:8]) + Z;
plot([-8:0.01:8], X);
Example 1

**Definition.** Let $T$ be a subset of the real line, $T \subseteq \mathbb{R}$. A set of random variables $\{X_t, t \in T\}$ is called a random (or stochastic) process with time domain $T$.

**Definition.** Let $\{X_t, t \in T\}$ be a random process. The function $m_{X}: T \mapsto \mathbb{R}$ given by $m_X(t) = E[X_t]$ is called the mean function of the random process $\{X_t, t \in T\}$ (if the expectation exists).

**Definition.** Let $\{X_t, t \in T\}$ be a random process. The function $r_{X}: T^2 \mapsto \mathbb{R}$ given by $r_X(s, t) = C[X_s, X_t]$ is called the covariance function of the random process $\{X_t, t \in T\}$ (if the covariance exists).

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Is this process weakly stationary?

Let $A \in \mathcal{U}(0, 1)$ and $\phi \in \mathcal{U}(0, 2\pi)$ be two independent random variables. Define the random process $\{X(t), t \in \mathbb{R}\}$ by:

$$X(t) = A \cos(t + \phi).$$

Is this process weakly stationary?

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Is this process weakly stationary?

$$X(t) = Y \sin(t) + Z.$$
Let $A \in U(0, 1)$ and $\phi \in U(0, 2\pi)$ be two independent random variables and let $\omega$ be a constant. Define the random process $\{X(t), t \in \mathbb{R}\}$ by:

$$X(t) = A \cos(\omega t + \phi).$$

Is this process weakly stationary?
Example II

\[ X(t) = A \cos(\omega t + \phi). \]
Example II

Let $T$ be a subset of the real line, $T \subseteq \mathbb{R}$. A set of random variables \{\(X_t, t \in T\)\} is called a random (or stochastic) process with time domain $T$.

Definition. Let \(\{X_t, t \in T\}\) be a random process. The function \(m_{X}: T \rightarrow \mathbb{R}\) given by \(m_{X}(t) = \mathbb{E}[X_t]\) is called the mean function of the random process \(\{X_t, t \in T\}\) (if the expectation exists).

Definition. Let \(\{X_t, t \in T\}\) be a random process. The function \(r_{X}: T^2 \rightarrow \mathbb{R}\) given by \(r_{X}(s, t) = \mathbb{C}[X_s, X_t]\) is called the covariance function of the random process \(\{X_t, t \in T\}\) (if the covariance exists).

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Let $Y$ and $Z$ be two independent random variables with expectation zero and variance $\sigma_Y^2$ and $\sigma_Z^2$. We construct the random process \(\{X(t), t \in \mathbb{R}\}\) by:

\[X(t) = Y \sin(t) + Z.\]

Is this process weakly stationary?

Let $A \in U(0,1)$ and $\phi \in U(0,2\pi)$ be two independent random variables and let $\omega$ be a constant. Define the random process \(\{X(t), t \in \mathbb{R}\}\) by:

\[X(t) = A \cos(\omega t + \phi).\]

Is this process weakly stationary?

Let $r$ be the covariance function of a weakly stationary random process. Prove that \(r(0) \geq |r(\tau)| \quad \forall \tau\).

\[X(t) = A \cos(\omega t + \phi).\]
## Properties of the Covariance Function

| General Processes | Definition. Let \( \{X(t), t \in T\} \) be any random process. The function  
|                   | \[ r_X(s, t) = C [X(s), X(t)] \]  
|                   | is called the covariance function. |
|                   | \[ r_X(\tau) = C [X(t), X(t + \tau)] \]  
|                   | is called the covariance function. |
Let $f_1, f_2, \ldots, f_n$ be positive real numbers. Let $A_1, A_2, \ldots, A_n$ be independent r.v. and let $\phi_1, \phi_2, \ldots, \phi_n$ be independent r.v. with uniform distribution in $(0, 2\pi]$. Define a random process, \( \{X(t), t \in \mathbb{R}\} \) by

\[
X(t) = \sum_{k=1}^{n} A_k \cos(2\pi f_k + \phi_k).
\]

Then,

\[
r_X(\tau) = \frac{1}{2} \sum_{k=1}^{n} \mathbb{E}[A_k^2] \cos(2\pi f_k \tau).
\]
Let $f_1, f_2, \ldots, f_n$ be positive real numbers. Let $A_1, A_2, \ldots, A_n$ be independent r.v. and let $\phi_1, \phi_2, \ldots, \phi_n$ be independent r.v. with uniform distribution in $(0, 2\pi]$. Define a random process, 

$$X(t) = \sum A_k \cos(2\pi f_n + \phi_k)$$

Then,
The Poisson Process

Definition. Let \( T \) be a subset of the real line, \( T \subseteq \mathbb{R} \). A set of random variables \( \{X_t, t \in T\} \) is called a random (or stochastic) process with time domain \( T \).

Definition. Let \( \{X_t, t \in T\} \) be a random process. The function \( m_{X} : T \mapsto \mathbb{R} \) given by \( m_{X}(t) = \mathbb{E}[X_t] \) is called the mean function of the random process \( \{X_t, t \in T\} \) (if the expectation exists).

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The Poisson Process

\[ \lambda = 2 \quad \lambda = 1 \quad \lambda = 0.5 \]
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Difference between two independent Poisson Processes

Let \( \{Y(t), t \geq 0\} \) and \( \{Z(t), t \geq 0\} \) be two independent Poisson processes and define \( X(t) = Y(t) - Z(t) \).
Difference between two independent Poisson Processes
Difference between two independent Poisson Processes
Let \( r \) be the covariance function of a weakly stationary random process. Prove that

\[
r(0) \geq |r(\tau)| \quad \forall \tau.
\]